JOURNAL OF APPLIED PHYSICS VOLUME 30, NUMBER 9 SEPTEMBER, 1959

Regular Articles

Use of the Principles of Conservation of Energy and Momentum in Connection with the Operation of Wave-Type Parametric Amplifiers

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(Received December 19, 1958)

The operation of a wave-type parametric amplifier may be thought of in terms of the force exerted by traveling discontinuities or reflecting elements on the waves present in the device. By invoking the principles of conservation of energy and momentum it is possible to understand in very simple terms certain limitations on the operation of such devices and to arrive very easily at certain general relations governing their behavior, including the Manley-Rowe relation. It is proposed that for a guided wave the momentum per unit distance is the power divided by the product of the group and phase velocities.

THE reflection of radiation from a moving piston has an honorable history in theoretical physics, and provides the standard derivation of Wien's displacement law.[1] In an unpublished work, G. C. Southworth and also R. Kompfner have proposed the use of reflection of microwaves from a virtual piston produced by the action of another electromagnetic wave on a nonlinear medium. This paper shows how it is possible, by using the conservation of energy and momentum in connection with the reflection of a wave from a moving discontinuity, to understand in very simple terms the operation of wave-type parametric amplifiers.[2-6]

Let us consider first the situation illustrated in Fig. 1. At the left, a completely reflecting wall is moving to the right with a velocity v. At a reference plane to the right, a plane electromagnetic (nondispersive) wave of power P_1 and frequency ω_1 moves to the left, while a wave of power P_2 and frequency ω_2 moves to the right.

The momentum per unit distance of a plane electromagnetic wave is P/c^2 and the momentum transfer per unit time is P/c.* The momentum of the power P_1 is negative and that of P_2 is positive. Accordingly, the conversion of P_1 to P_2 involves an increase in momentum at a rate

$$(1/c)(P_1+P_2). \tag{1}$$

The net momentum stored between the reflector and the reference plane is

$$(L/c^2)(-P_1+P_2). \tag{2}$$

As $dL/dt=-v$, the rate of increase of this momentum is

$$(v/c^2)(P_1-P_2). \tag{3}$$

The total rate of increase of momentum must be equal to f, where f is the force the reflecting plane exerts on the waves. Accordingly,

$$f=(1/c)(P_1+P_2)+(v/c^2)(P_1-P_2). \tag{4}$$

Let us now consider the energy. The net rate of energy flow to the right past the reference plane is

$$P_2-P_1. \tag{5}$$

The energy stored between the reflecting plane and the reference plane is

$$(L/c)(P_2+P_1). \tag{6}$$

The rate of change of this energy is

$$-(v/c)(P_2+P_1). \tag{7}$$

The total rate of increase of wave energy must be equal to fv, so that

$$fv=(P_2-P_1)-(v/c)(P_2+P_1). \tag{8}$$

From (4) and (8) we easily obtain

$$\frac{P_2}{P_1}=\frac{(1+v/c)^2}{(1-v/c)^2}. \tag{9}$$

Let us now consider a necessary relation between the frequencies involved. We will assume that the fields of the two waves vary with time and distance as

$$\cos\omega_1(t+z/c), \quad \cos\omega_2(t-z/c). \tag{10}$$

At the piston, which moves to the right with a velocity v, we will observe a variation of the fields of the

FIG. 1. A plane nondispersive electromagnetic wave reflected from a piston moving to the right with a velocity v.

[1] For instance, in F. K. Richtmyer and E. H. Kennard, *Introduction to Modern Physics* (McGraw-Hill Book Company, Inc., 1942), third edition.
[2] A. L. Cullen, Nature **181**, 332 (1958).
[3] P. K. Tien and H. Suhl, Proc. Inst. Radio Engrs. **46**, 700–706 (1958).
[4] W. H. Louisell and C. F. Quate, Proc. Inst. Radio Engrs. **46**, 700–716 (1958).
[5] Bell Labs. Record **36**, 250 (1958).
[6] R. S. Engelbrecht, Proc. Inst. Radio Engrs. **46**, 1655 (1958); also presented at the Institute of Radio Engineers Conference on Electron Devices, June, 1958.
* This is discussed in the Appendix.

two waves with time as

$$\cos\omega_1(1+v/c)t, \quad \cos\omega_2(1-v/c)t. \tag{11}$$

The frequencies of the two waves at the piston must be the same; that is,

$$|\omega_1(1+v/c)| = |\omega_2(1-v/c)|. \tag{12}$$

This is mathematically so if

$$v>c, \quad \frac{\omega_2}{\omega_1}=\frac{v/c+1}{v/c-1}. \tag{13}$$

A more realistic case is

$$v<c, \quad \frac{\omega_2}{\omega_1}=\frac{1+v/c}{1-v/c}. \tag{14}$$

We note from (9) that

$$(P_2/P_1)=(\omega_2/\omega_1)^2. \tag{15}$$

We may be inclined to compare this with the Manley-Rowe relationship.[7] This relationship holds for purely reactive devices involving frequency conversion. It relates to the net input and output powers at different frequencies. For a device with a net input P_{I} at frequency ω_{I} and a net output power P_{II} at frequency ω_{II} the Manley-Rowe relationship is

$$(P_{\mathrm{II}}/P_{\mathrm{I}})=(\omega_{\mathrm{II}}/\omega_{\mathrm{I}}). \tag{16}$$

We see that for the device we have analyzed the Manley-Rowe relation does not hold, nor should it. Part of the phenomena analyzed involves a steady decrease in energy stored in the system, while the Manley-Rowe relationship applies to a stationary process in which the energy storage does not steadily increase or decrease with time.

Let us now consider a wave guide or transmission line into which are inserted a number of discontinuities, perhaps little paddles, all moving to the right with a velocity v, as shown in Fig. 2. As we shall see later, these can be so spaced as to give a cumulative reflection.

Such a sequence of paddles of course has an effect analogous to the varying reactances actually used in parametric amplifiers.[2–6]

Let us now assume that we have achieved such a condition, and that the stored energy and momentum can be considered to be constant in the device of Fig. 2. Then the only change in energy and momentum caused by the force on the waves is the excess of energy and mementum of P_2 over P_1. If we are dealing with plane

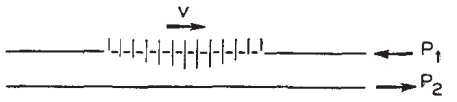

FIG. 2. A wave reflected by a series of moving paddles so spaced that the reflections add in phase.

[7] J. M. Manley and H. E. Rowe, Proc. Inst. Radio Engrs. 44, 904–913 (1956).

electromagnetic waves, the rate of increase of momentum, which is equal to the force, is

$$f=(1/c)(P_2+P_1). \tag{17}$$

The rate of increase of energy, which is equal to fv, is

$$fv=(P_2-P_1). \tag{18}$$

From (17) and (18) we obtain

$$\frac{P_2}{P_1}=\frac{1+v/c}{1-v/c}. \tag{19}$$

We see that (13) is inconsistent with this, for it would make the ratio of two positive quantities, P_1 and P_2, negative. Thus, v must be less than c, and (14) must apply. From (19) and (14) we obtain

$$(P_2/P_1)=(\omega_2/\omega_1). \tag{20}$$

This exemplifies the Manley-Rowe relation, given in (16).

Let us return to the spacing of paddles or reflectors necessary to attain this sort of operation. We must make the spacing l between paddles such that the relative phase change between the two waves in going the distance l from one paddle to the next is $2n\pi$, where n is an integer.

In this case, in which we have two waves with phase constants ω_1/c and ω_2/c traveling in opposite directions, this means that†

$$\frac{\omega_2 l}{c}+\frac{\omega_1 l}{c}=2n\pi \tag{21}$$

$$f_1+f_2=\frac{nc}{l}. \tag{22}$$

The frequency f_p with which paddles enter the guide is

$$f_p=v/l. \tag{23}$$

From (14) we obtain

$$\frac{c}{v}=\frac{f_2+f_1}{f_2-f_1}. \tag{24}$$

Accordingly, by using (23) and (24), (22) can be written

$$f_2-f_1=nf_p. \tag{25}$$

Thus, the paddle or "pump" frequency, or a harmonic of this frequency, must be equal to the difference between the output and input frequencies.

So far we have considered the case of a plane electromagnetic wave, for which the electromagnetic momentum per unit length is the power divided by c^2 and the momentum flow per unit time is the power divided by c. Let us now consider the case of a dispersive wave with

† Here we have disregarded any effect of the presence of the paddles on the wave velocity.

a phase velocity v_f and a group velocity v_g as illustrated in Fig. 3. Let us assume that the momentum per unit length, p, is a factor k times the power P

$$p=kP. \tag{26}$$

The momentum flow will then be the group velocity v_g times the momentum per unit length, p.

As the wave to the right has a frequency ω_2 different from the frequency ω_1 of the wave to the left, so it will have a phase velocity v_{f2} different from the phase velocity v_{f1} of the wave to the left. Thus, the argument which led to (14) now leads to

$$\frac{\omega_2}{\omega_1}=\frac{1+v/v_{f1}}{1-v/v_{f2}}. \tag{27}$$

The momentum equation becomes

$$f=v_{g2}k_2P_2+v_{g1}k_1P_1. \tag{28}$$

The energy equation is

$$fv=P_2-P_1. \tag{29}$$

From (28) and (29) we obtain

$$\frac{P_2}{P_1}=\frac{1+vv_{g2}k_2}{1-vv_{g1}k_1}. \tag{30}$$

We see from (27) and (30) that the Manley-Rowe relation will apply in this case if for each wave the factor k is

$$k=(1/v_gv_f) \tag{31}$$

so that the momentum per unit length p is

$$p=(P/v_gv_f). \tag{32}$$

The writer believes that (32) must apply for any guided wave whether it is an electromagnetic wave or not.

So far we have considered illustrative rather than interesting examples. Let us now consider the case of the parametric wave amplifier shown in Fig. 4. In order to simplify the treatment we will assume nondispersive electromagnetic waves of frequencies ω_1 and ω_2 traveling to the right with velocities c_1 and c_2. The first wave enters with a power P_1 and emerges with a power P_1'. The second wave enters with a power P_2 and emerges with a power P_2'.

For this system the momentum equation is

$$f=(1/c_1)(P_1'-P_1)+(1/c_2)(P_2'-P_2). \tag{33}$$

The energy equation is

$$fv=(P_1'-P_1)+(P_2'-P_2). \tag{34}$$

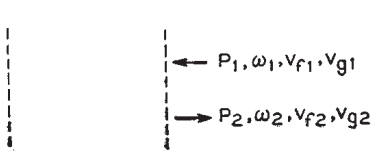

FIG. 3. The reflection by moving paddles of a dispersive wave whose phase velocity v_f and whose group velocity v_g are functions of the radiation frequency ω.

FIG. 4. Power flow at two frequencies in a parametric amplifier involving two waves having phase and group velocities to the right and carrying power to the right.

These yield

$$\frac{(P_1'-P_1)}{(P_2'-P_2)}=-\frac{(1-v/c_2)}{(1-v/c_1)}. \tag{35}$$

If the frequencies of the two waves are to be the same at the moving paddles we must have

$$|\omega_1(1-v/c_1)|=|\omega_2(1-v/c_2)|. \tag{36}$$

We see from (35) that if both waves are to gain or lose power, as in the case of a growing or attenuated wave, v must lie between c_1 and c_2. Hence the appropriate solution of (36) is

$$\frac{\omega_2}{\omega_1}=\frac{v/c_1-1}{1-v/c_2}. \tag{37}$$

With respect to the moving paddles, the first wave, which has a phase constant ω_1/c_1, moves to the left with respect to the paddle and the second wave, with a phase constant ω_2/c_2, moves to the right with respect to the paddle. Hence, to make the relative phase of the fields change by an amount $2n\pi$ in going a distance l from one paddle to the next we must have

$$(\omega_1l/c_1)+(\omega_2l/c_2)=2n\pi. \tag{38}$$

By using (37) and (23) we can write this

$$f_1+f_2=nf_p. \tag{39}$$

We should note from (35) and (37) that

$$\frac{P_1'-P_1}{P_2'-P_2}=\frac{f_1}{f_2}. \tag{40}$$

This is in accord with the Manley-Rowe relationship for this case.

If we regard ω_2 as the idler frequency, we can make P_2' the idler power at the input zero, so that the signal power P_1 will be the only input required. At the output, however, we necessarily have power at the idler frequency ω_2 as well as at the signal frequency ω_1. This is necessarily so because we cannot simultaneoulsy satisfy both the conservation of energy and the conservation of momentum with one electromagnetic wave. However, we should note that as the velocity of the moving reflectors approaches the velocity c_1 of the signal wave, (and as f_2 approaches zero), the idler power P_2' becomes very small compared with the amplified signal power P_1', and approaches zero as v approaches c_1.

This parametric amplifier deserves some further consideration. It is clear that to get a large gain one can

FIG. 5. A parametric amplifier involving a fast space charge wave with a positive power flow and a slow space charge wave with a negative power flow.

either increase the magnitude of the reflection at the discontinuities or increase the number of moving reflectors. Increasing the magnitude of the reflection will tend to increase the gain in proportion to the total amount of reflection. Increasing the number of reflectors will tend to increase the ratio of output power to input power exponentially with the number of reflectors, as we may regard successive reflectors as providing gain in tandem.

In the example given above, two nondispersive modes having velocities c_1 and c_2 are assumed. The same sort of analysis can be used in the case of a single dispersive mode, for which the phase velocities at the two frequencies lie above and below the paddle velocity.

In the case of two forward waves of different frequencies in a nondispersive circuit, differences in velocity due to the effect of the paddles would be very important, and as this has not been taken into account, the analysis cannot be applied in its present form to this case.

Parametric amplification of space-charge waves on an electron beam provides an interesting example closely analogous to that given above. This system is illustrated in Fig. 5. Here we have at the left a fast input wave of power P_f, and a slow input wave of power $-P_s$, and at the right we have a fast wave to the right of power P_f', a slow idler wave of power $-P_s'$.

It is shown in the Appendix that for a space-charge wave the rate of momentum transfer is given in terms of the power by

$$(\beta v_g/\omega u_0)P. \tag{41}$$

If we assume that the effective plasma frequency does not vary with frequency then

$$v_g = u_0. \tag{42}$$

We can also write

$$\beta = (\omega/v_f). \tag{43}$$

Thus, disregarding the variation of effective plasma frequency with frequency, we can write the rate of momentum transfer as

$$P/v_f. \tag{44}$$

The momentum equation is

$$f = (1/v_{ff})(P_f'-P_f) - (1/v_{fs})(P_s'-P_s). \tag{45}$$

Here v_{ff} and v_{fs} are the phase velocities of the fast and slow waves, respectively.

The energy equation is

$$fv = (P_f'-P_f) - (P_s'-P_s). \tag{46}$$

From (45) and (46) we obtain

$$\frac{(P_s'-P_s)}{(P_f'-P_f)} = \frac{1-v/v_{ff}}{1-v/v_{fs}}. \tag{47}$$

We can easily show that

$$\frac{f_s}{f_f} = \frac{1-v/v_{ff}}{1-v/v_{fs}} \tag{48}$$

$$f_f + f_s = nf_p. \tag{49}$$

We observe that the Manley-Rowe relationship holds. We further observe that if we let the velocity v of the moving reflections approach the velocity v_{ff} of the fast wave, so that f_s approaches zero, we can have a large fast-wave power with a very small slow-wave power.

The purpose of the foregoing work has been to relate known phenomena and formulas concerning parametric amplifiers to simple considerations of conservation of energy and momentum. The chief weakness of this work is that nothing has been done to relate the physical nature or properties of the moving discontinuities or reflections to the power ratios. Perhaps something of this sort could be accomplished.

ACKNOWLEDGMENT

The writer is indebted to P. K. Tien for valuable comments and suggestions.

APPENDIX. MOMENTUM IN ELECTRON FLOW

Maxwell's equations yield a relation involving Poynting's vector P_e and the electromagnetic stored energy W_e

$$\nabla \cdot P_e + \partial/\partial t(W_e) + E \cdot J = 0. \tag{1A}$$

By using this we can find analogous kinetic power and energy quantities for a one-dimensional electron beam.[8]

There is a relation similar to (1A) concerning electromagnetic momentum[9]

$$\text{div}(-T) + \epsilon\mu\partial/\partial t(E \times H) + \rho E + J \times B = 0. \tag{2A}$$

Here J is convection current; thus, the terms ρE and $J \times B$ are force terms. The quantity

$$\epsilon\mu E \times H = \epsilon\mu P_e \tag{3A}$$

is usually interpreted as the electromagnetic momentum, p_e. Thus, let us consider a case in which ρE and $J \times B$ are either zero or else have no component in the direction under consideration. In this case we can write (2A) as

$$\text{div}(-T) + \partial p_e/\partial t = 0. \tag{4A}$$

[8] W. H. Louisell and J. R. Pierce, Proc. Inst. Radio Engrs. **43**, 425–427 (1955).

[9] J. C. Slater, *Microwave Electronics* (D. Van Nostrand, Inc., Princeton, New Jersey, 1950), Chap. 3.

The quantity T is a tensor

$$T_{xx}=\epsilon/2(E_x^2-E_y^2-E_z^2)+\mu/2(H_x^2-H_y^2-H_z^2)$$

similarly for T_{yy}, T_{zz}

$$T_{xy}=T_{yx}=\epsilon E_xE_y+\mu H_xH_y \tag{5A}$$

similarly for T_{yz}, T_{zx}.

The divergence of T is a vector; the x component of this vector is

$$(\mathrm{div}T)_x=\partial T_{xx}/\partial x+\partial T_{xy}/\partial y+\partial T_{xz}/\partial z. \tag{6A}$$

The tensor T is usually interpreted as the stress in the medium. This interpretation is natural in electrostatic and magnetostatic problems. The interpretation seems odd in dealing with electromagnetic waves in vacuum, in which ρ and J are zero. Imagine, for instance, that a radar pulse passes us. It has momentum, so that p_e rises and falls. T also changes. How are we to interpret this? We can if we wish think of $-T$ as being the momentum flow, so that $\mathrm{div}(-T)$ is the rate at which momentum flow increases with distance; thus, if momentum is to be conserved we must have

$$\partial p_e/\partial t=-\mathrm{div}(-T). \tag{7A}$$

We see that $(-T)$ must be a tensor, for momentum itself is a vector, and we can think of the x, y, and z components of momentum as traveling with the wave in the x direction across the yz plane.

We should note, however, that we can have an average value of momentum flow without having an average value of momentum. Thus, a wave traveling in the $+x$ direction has a positive momentum and a positive velocity, and the average momentum flow in the x direction is positive, while a wave traveling in the opposite direction has a negative momentum, a negative velocity and a positive component of momentum flow in the x direction. If two such waves are simultaneoulsy present, we can have a positive average momentum flow in the x direction without any average momentum in the x direction.

In dealing with guided waves we wish to deal with the total momentum in the direction of propagation. Thus, we should consider only the component of momentum in the direction of propagation, and we should integrate the quantities of (2A) with respect to the coordinates normal to the direction of propagation. We will find, however, that in waves guided by metallic circuits the components of ρE and $J\times B$ in the direction of propagation are not necessarily zero. In a perfectly conducting wave guide they are, for at the walls where ρ and J are not zero there is no component of E in the direction of propagation and no component of B normal to the wall. Thus, in the case of the wave guide we can take $\epsilon\mu P_e$ as equal to p_e, the electromagnetic momentum. This is not so for some other circuits, including the helix.

It seems clear from application of the Manley-Rowe relation that the general expression for the momentum p_e of a guided wave in terms of the power P_e must be

$$p_e=(P_e/v_gv_f). \tag{8A}$$

Here v_g and v_f are the group and phase velocities of the wave. This is consistent with the foregoing discussion. For in a waveguide, for instance,

$$v_g=c[1-(f_c/f)^2]^{\frac{1}{2}} \tag{9A}$$

$$v_f=c[1-(f_c/f)^2]^{-\frac{1}{2}} \tag{10A}$$

$$v_gv_f=c^2. \tag{11A}$$

Here f_c is the cutoff frequency of the wave guide.

Let us now return to the case in which the parts of ρE and $J\times B$ associated with the electromagnetic wave itself have no components in the direction of propagation, which we will take as the z direction. We can then take P_e/c^2 as the electromagnetic momentum per unit length, p_e. Further, let us consider that there are electrons in the electromagnetic field which are acted on by an electric field in the z direction only, and which move only in that direction. Newton's laws of motion tell us that

$$E_z=-(m/e)(\partial v/\partial t+v\partial v/\partial z). \tag{12A}$$

In this case the current J normal to the beam is zero, and there are no magnetic forces in the z direction. Accordingly

$$(\mathrm{div}T_e)_z=\rho E_z+\partial/\partial t(p_e)_z. \tag{13A}$$

We wish to associate with the electron flow a momentum flow $-T_k$ and a kinetic momentum p_k such that

$$[\mathrm{div}(T_e+T_k)]_z=\partial/\partial t(p_e+p_k). \tag{14A}$$

We can easily verify that the following quantities used in (14A) lead, with the aid of (12A), to (13A):

$$-T_k=-(m/e)Jv \tag{15A}$$

$$p_k=-(m/e)\rho v. \tag{16A}$$

Here

$$J=\rho v \tag{17A}$$

and we have the continuity relation

$$\partial\rho/\partial t=-\partial J/\partial z. \tag{18A}$$

Suppose that we now write linear equations which we say apply to a system whose performance is similar to that of the electron stream at low levels. This linear system obeys the relations

$$E_z=-(m/e)(\partial v/\partial t+u_0\partial v/\partial z) \tag{19A}$$

$$J=J_z=-\rho_0v+u_0\rho. \tag{20A}$$

Here v and o are ac quantities and u_0 and ρ_0 are constants; we identify u_0 with dc electron velocity and $-\rho_0$ with dc charge density. The divergence relation is now

$$\partial\rho/\partial t=\rho_0\partial v/\partial z-u_0\partial\rho/\partial z. \tag{21A}$$

We find that the following quantities together with (17A)–(21A) lead to (14A)

$$-T_k=-(m/e)(-\rho_0u_0^2-\rho_0u_0v-\tfrac{1}{2}\rho_0v^2+u_0ov) \quad (22A)$$

$$p_k=-(m/e)(-\rho_0u_0-\rho_0v+\rho v). \quad (23A)$$

This seems a strange assortment of first-order and second-order terms.

Let us now consider the case of a space-charge wave. We can write the charge density ρ in terms of the convection current;

$$\rho=(\beta/\omega)i. \quad (24A)$$

The kinetic voltage U is

$$U=-u_0v/(e/m) \quad (25A)$$

so that

$$v=-(e/m)(U/u_0). \quad (26A)$$

Thus, the only term of (23A) which can have an average third on the right, will have an average value

$$(\beta/\omega u_0)\tfrac{1}{2}(i^*U+iU^*)=(\beta/\omega u_0)P. \quad (27A)$$

We may express the phase constants of a space-charge wave by

$$\beta=(\omega/u_0)\pm(\omega_q/u_0)=(\omega/v_f). \quad (28A)$$

Here the + sign applies for the slow, negative-energy wave, and the − sign for the fast, positive-energy wave.

If we assume that ω_q does not change with frequency,

$$v_g=(\partial\beta/\partial\omega)^{-1}=u_0. \quad (29A)$$

Hence, according to this approximation, we obtain from (22A), (28A), and (29A) that the momentum per unit length is

$$P/u_0v_f. \quad (30A)$$

We note that (29A) and (30A) are consistent with (8A). Accordingly, the rate of momentum transfer is this quantity times the group velocity u_0, or

$$P/v_f. \quad (31A)$$